Pergamon

Int. J. Solids Structures Vol. 34, Nos 31–32, pp. 4073–4086, 1997

0020–7683/97 $17.00 + .00

PII: S0020–7683(97)00014–0

ROLLING COMPLIANCE FOR ELASTIC AND VISCO-ELASTIC CONFORMING BINDER CONTACT

HAN ZHU
Applied Research Associates, Inc., Tyndall Air Force Base, Florida 32403, U.S.A.

CHING S. CHANG
Department of Civil and Environmental Engineering, University of Massachusetts, Amherst MA 01003, U.S.A.

and

JEFF. W. RISH III
Pavements and Facilities Section, Wright Laboratory, Tyndall Air Force Base, Florida 32403, U.S.A.

(*Received* 8 *November* 1995; *in revised form* 11 *December* 1996)

Abstract—This article presents a method of analysis for deriving expressions of the rolling compliance for a system of two elastic particles bonded by a thin layer of elastic or visco-elastic binder. The governing equations of rolling compliance for this system are Fredholm integral equations of the second type for which it is very difficult to find closed-form solutions. The method of analysis is proposed to derive closed-form compliances in the form of the upper and lower bounds. The best estimate of rolling compliance is then derived based on the upper and lower bound solutions. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The subject of layer/binder contact frequently occurs in the study of granular/particulate materials such as asphaltic concrete or cemented sand. This subject is also important in tribology, involving the mechanical behavior of coated materials. Many topics in this area have been investigated in the past years (for example, Muki, 1960, Goodman and Keer, 1975, Bentall and Johnson, 1968, Meijers, 1968, Alblas and Kuipers, 1970, Matthewson, 1981, Keer *et al.*, 1991). The normal and tangential compliances for two bonded particles in compression and sliding can be found in the work by Dvorkin *et al.* (1991, 1994) for elastic binder and in the work by Zhu *et al.* (1996) for visco-elastic binder.

Besides the modes of compression and sliding, rolling of particles is also frequently observed in particulate materials under a shear deformation. Therefore, modelling for the rolling compliance is a useful micro-mechanics study that can be used in analyzing the moment transmitting in a particulate material under shear deformation. In the past, most studies on rolling contact are related to cylinders rolling on metal plates (Johnson, 1985, Gladwell, 1980). In this situation, the rolling of cylinders is caused by a pull of the metal plate between cylinders. The driving force for the rolling of cylinders is the frictional sliding force developed at the interfacial surface. There is no force-couple at the interfacial surface. However, in the present problem, the mechanism of rolling is very different from the kinetic rolling of the cylinders due to the pull-out of a metal plate. The rolling of two particles generates deformation of the particle-binder system and develops a force-couple at the interfacial surface. Very few studies can be found on the modelling this type of rolling mechanism.

$$L^*(r,\rho,\theta,\phi) = \sqrt{(r\cos\theta-\rho\cos\phi)^2+(r\sin\theta-\rho\sin\phi)^2}. \tag{7}$$

Substituting eqns (5)–(7) into eqn (4), it follows

$$\begin{aligned}\omega x &= w_1(r,\theta)+w_2(r,\theta)\\ &= h(r)\frac{p^*(r,\theta)}{E_2}+\frac{1-v_1^2}{\pi E_1}\int_0^a\int_0^{2\pi}\frac{p^*(\rho,\phi)\rho\,\mathrm{d}\phi\,\mathrm{d}\rho}{L^*(r,\rho,\theta,\phi)}.\end{aligned} \tag{8}$$

We further utilize the concept of Fourier series expansion and substitute $p^*(\rho,\phi)$ with $p(\rho)\cos\phi$ to eqn (8) (see details in Kanwal's book (1971)). After a few steps of manipulation, it yields the relationship between ω-$p(r)$, given by:

$$\omega r = \frac{h(r)p(r)}{E_2}+\frac{1-v_1^2}{\pi E_1}\int_0^a\int_0^{2\pi}\frac{p(\rho)\cos\phi\rho\,\mathrm{d}\phi\,\mathrm{d}\rho}{L(r,\rho,\phi)}$$

$$L(r,\rho,\phi) = \sqrt{(r^2+\rho^2-2r\rho\cos\phi)}. \tag{9}$$

In eqn (9), the interfacial pressure, $p(\rho)\cos\phi$, is related to the resultant rolling couple M by:

$$M = \int_0^a\int_0^{2\pi} rp^*(r,\theta)\cos\theta r\,\mathrm{d}\theta\,\mathrm{d}r = \pi\int_0^a p(r)r^2\,\mathrm{d}r. \tag{10}$$

It can be seen that eqns (9) and (10) indirectly provide the compliance relationship between the relative angular movement ω and the contact movement M through the interfacial pressure function $p(r)$.

3. SOLUTIONS FOR ROLLING COMPLIANCE

3.1. *Existing solutions for two limiting cases*

The exact solutions of the interfacial pressure function $p(r)$ in eqn (9) are known for two limiting cases, namely (1) rigid particle case (i.e., $E_1 \to \infty$ while E_2 is finite), and (2) rigid binder case (i.e., E_1 is finite while $E_2 \to \infty$). The rolling compliance relationship under these two extreme conditions are described in this section.

Rigid particle case. In the rigid particle case, the relative angular movement of the two contact bodies is contributed only from the deformation of binder. Thus $p(r)$ can be easily determined by (denoted as $p_1(r)$):

$$p_1(r) = \frac{E_2 C_r' r M}{h(r)};\quad C_r' = \frac{2h_0 d^2}{a^4\pi E_2[d-\ln(1+d)]} \tag{11}$$

where d is the shape parameter defined in eqn (2).

Substituting the interfacial pressure function $p_1(r)$ into eqn (9), the corresponding rolling contact compliance is obtained, given by:

$$\omega = C_r' M. \tag{12}$$

Rigid binder case. For the case of rigid binder, the exact solution of $p(r)$ is known (Johnson, 1985) (denoted as $p_2(r)$) and it reads in the range of $0 \leqslant d < 1$. For a planar interfacial surface, d is zero. For a spherical interfacial surface, d is given by

$$d = \frac{a^2}{2Rh_0} \tag{2}$$

where R is the radius of the spherical particles. Typical shapes of the interfacial surface are shown in Fig. 1 for $d = 0$ and $d = 0.2$.

Material properties are denoted by shear moduli G_1 and G_2 and Poisson's ratios v_1 and v_2 where the subscripts 1 and 2 represent the particle and the binder, respectively. The Young's modulus E_1 and E_2 are defined as

$$E_i = 2G_i(1+v_i)\,; \quad i = 1,2. \tag{3}$$

In the particle-binder system, as schematically shown in Fig. 1, both the top and bottom planes of the particles rotate a degree ω above the y-axis. This rolling motion results in a force couple M about y-axis of the system. The objective of this paper is to establish a relationship between the force couple M and the relative angular movement ω of the two particles.

The relative angular movement of two particles is developed from deformation of both the particles and the binder. The bonding condition is assumed to be perfect, i.e., continuity of displacement and traction at the interfaces between particles and the binder is implied. Since the displacement in the x-direction are anti-symmetric about the $x = 0$ plane, the angular movement ω must be contributed from the displacements in the z-direction. Since the system is symmetric about the $z = 0$ plane, the $z = 0$ plane remains immobile and the relative angular movement ω rotating along the y-axis can be separated into two components, i.e.,

$$\omega x = w_1(r,\theta) + w_2(r,\theta) \tag{4}$$

where $w_1(r,\theta)$ is the displacement in the z-direction at the binder-particle interface (i.e., at $z = h(r)$) relative to that at the top plane, and $w_2(r,\theta)$ is the displacement in the z-direction at the binder-particle interface relative to that at the $z = 0$ plane.

We introduce the assumption of a thin layer binder such that we can approximate the normal strain to be uniform in the z direction across the binder. Let $z = 0$ be a plane of symmetry and the binder normal displacement vanishes at $z = 0$. Thus the displacement in z-direction $w_2(r,\theta)$ can be expressed as follows:

$$w_2(r,\theta) = h(r)\frac{p^*(r,\theta)}{E_2} \tag{5}$$

where $p^*(r,\theta)$ is the interfacial normal pressure between the particle and the binder.

We also assume that the characteristic dimension of the particle is much larger than that of the particle-binder contact area. Thus, it is justifiable to pursue the analysis of $w_1(r,\theta)$ based on a half-space premise. Following the well-known Boussinesq's equation, $w_1(r,\theta)$ can be related to $p^*(r,\phi)$ by:

$$w_1(r,\theta) = \frac{(1-v_1^2)}{\pi E_1}\int_0^a\int_0^{2\pi}\frac{p^*(\rho,\phi)\rho\,\mathrm{d}\phi\,\mathrm{d}\rho}{L^*(r,\rho,\theta,\phi)} \tag{6}$$

where

This paper is focused on the rolling compliance of a system of two elastic particles bonded by a layer of elastic or visco-elastic binder. We aim to derive closed-form relationships between the force couples at the inter-particle binder and the relative angular movement of the two particles. Closed-form expressions are of particular interest because they can be readily incorporated into discrete element methods for the analysis of a large assembly of particles.

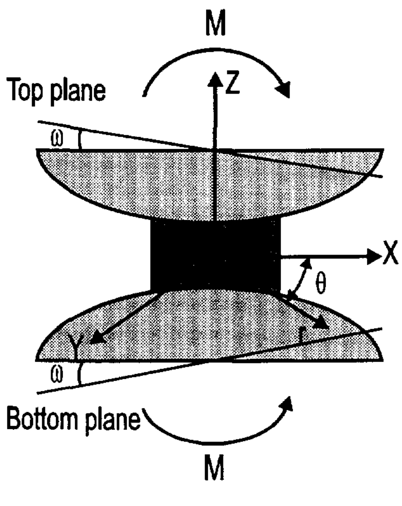
Progression of the article begins with establishing integral equations that govern the rolling of two particles bonded by a thin layer of binder. Since this equation is in a complicated integral form, it is difficult to obtain a solution. A method of analysis is proposed to derive the upper and lower bounds for the rolling compliance of the two particle system. Then, it is followed by the derivation of the best estimated rolling compliance.

2. FORMULATION OF THE PROBLEM

Figure 1 shows two identical particles bonded by a binder in a cartesian coordinate system. A cylindrical coordinate system is also shown in Fig. 1 since the configuration is axi-symmetric about the z-axis. The system is also symmetric about the $z = 0$ plane. Therefore, only upper half of the system shown in Fig. 1 is considered in the analysis. The interfacial surface between the particle and the binder is represented by the function $z = h(r)$, given by

$$h(r) = h_0\left(1+d\frac{r^2}{a^2}\right) \tag{1}$$

where a is the radius of contact area, h_0 is the thickness of the binder at $r = 0$. The dimensionless shape parameter, d, relating to the curvature of particle surface and is limited



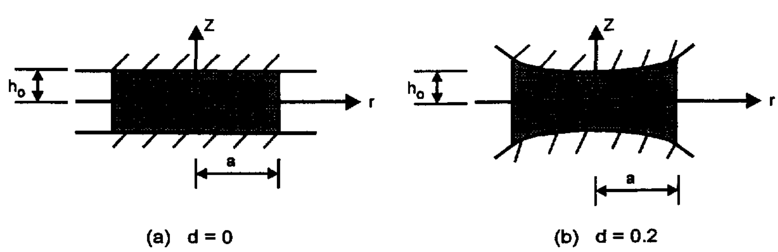


Fig. 1. Schematic plot of two particles bonded by a binder in a cartesian coordinate.

$$p_2(r) = \frac{3Mr}{2\pi a^3\sqrt{a^2-r^2}}. \tag{13}$$

Substituting the interfacial pressure function $p_2(r)$ into eqn (9), the rolling compliance relationship can be obtained accordingly:

$$\omega = C_r''M \tag{14}$$

where

$$C_r'' = \frac{3(1-\nu_1^2)}{4a^3E_1}. \tag{15}$$

Closed-form expressions for rolling compliance are existing for the two limiting cases. However, for finite E_1 and E_2 closed-form solutions to eqn (9) are difficult to obtain, especially when singularity is involved at the edge of the interface between binder and particle. Therefore, in what follows, we seek for the approximate solutions which represent the upper and lower bounds. The method of deriving approximate solutions does not involve the detailed solution of interfacial pressure. Thus, the solution procedure is greatly simplified.

3.2. *Upper bound solution*

For the purpose of deriving upper bound solution, we multiply $r^2/h(r)$ to eqn (9), integrate the equation over the range $0 \leqslant r \leqslant a$, and obtain:

$$\omega = C_r'M + C_r''\frac{8d^2}{3\pi[d-\ln(1+d)]}\int_0^a f(\rho)p(\rho)\rho^2\,\mathrm{d}\rho \tag{16}$$

where C_r' and C_r'' are the compliances given previously in eqns (12) and (14), and the function $f(\rho)$ is:

$$\begin{aligned} f(\rho) &= \int_0^1\int_0^{2\pi}\frac{\cos\phi r^2\,\mathrm{d}\phi\,\mathrm{d}r}{(1+r^2d)(\rho/a)\sqrt{r^2+(\rho/a)^2-2r(\rho/a)\cos\phi}} \\ &= \int_0^1\int_0^{\pi/2}\frac{8r^3\cos^2\phi\,\mathrm{d}\phi\,\mathrm{d}r}{(1+r^2d)L_+L_-(L_-+L_+)} \end{aligned} \tag{17}$$

and

$$\begin{aligned} L_+ &= \sqrt{r^2+(\rho/a)^2+2r(\rho/a)\cos\phi}; \\ L_- &= \sqrt{r^2+(\rho/a)^2-2r(\rho/a)\cos\phi}. \end{aligned} \tag{18}$$

We now have converted the original governing eqn (9) into the new governing eqn (16) which is in terms of the compliances of the two extreme conditions. This conversion does not make it easier for analytical evaluation because eqn (16) still contains integral of the unknown interfacial pressure function. However, this conversion has made it possible to utilize the monotonic property of $f(\rho)$ to pursue the upper bound analysis. Since $f(\rho)$ is

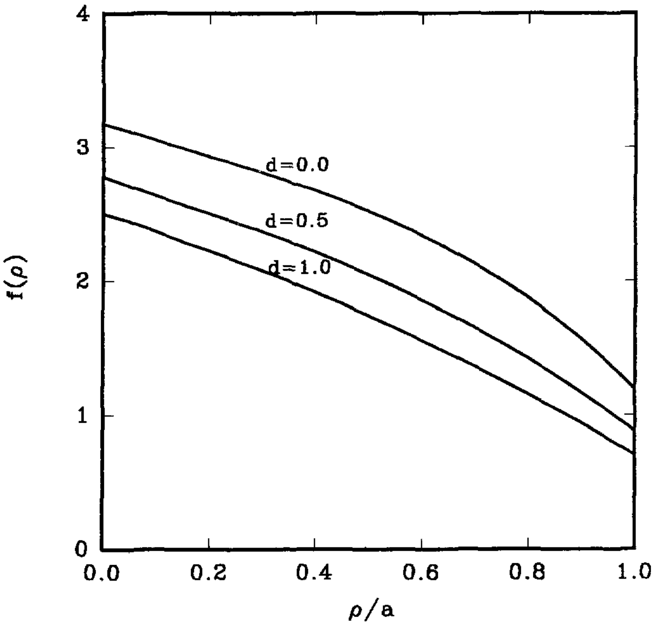


Fig. 2. The monotonically decreasing property of the function $f(\rho)$.

monotonically decreasing in the range of integration (see Fig. 2), it leads to the following inequality:

$$\int_0^a f(\rho)p(\rho)\rho^2\,\mathrm{d}\rho < \int_0^a f(0)p(\rho)\rho^2\,\mathrm{d}\rho = \frac{M}{\sqrt{d}}\tan^{-1}\sqrt{d}. \tag{19}$$

Note that the integral in eqn (19) is reduced to a simple algebraic form because the function $f(0)$ has a closed-form expression.

Substituting the inequality of eqn (19) into (16), the upper bound solution for the rolling compliance is derived as:

$$\omega < (C'_r + C''_r b_1)M \tag{20}$$

where

$$b_1 = \frac{8d^2 tg^{-1}\sqrt{d}}{3\pi[d-\ln(1+d)]\sqrt{d}} > 1. \tag{21}$$

It is noted that we develop the compliance relationship without solving the interfacial pressure distribution.

3.3. *Lower bound solution*

Similarly, we can utilize the monotonic property of $f(\rho)$ to pursue the lower bound analysis by replacing $f(\rho)$ with $f(a)$ in eqn (16). However, $f(a)$ does not have a closed-form expression. Therefore, we pursue the lower bound analysis from an alternative approach. We convert the original governing equation into a new governing equation using a different multiplier: We multiply $rp_2(r)$ to eqn (9); then integrate the equation with respect to the variable r over the range $0 < r < a$, which yields:

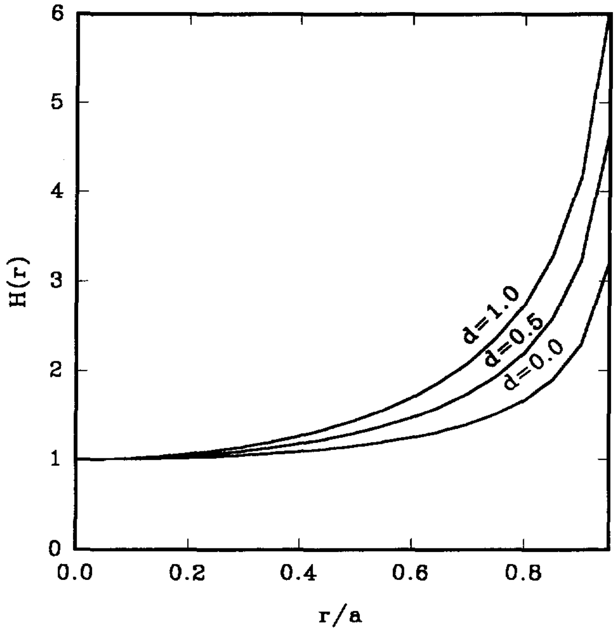


Fig. 3. The monotonically increasing property of the function $H(r)$.

$$\omega = C_r' \frac{3\pi a[d-\ln(1+d)]}{4h_0 d^2} \int_0^a p(r)H(r)r^2 \, dr + C_r'' M;$$

$$H(r) = \frac{h(r)}{\sqrt{a^2-r^2}}. \tag{22}$$

The new governing equation is also in terms of compliances of extreme conditions. It is easily seen that $H(r)$ in eqn (22) increase monotonically with respect to r in the range $0 \leqslant r \leqslant a$ as shown in Fig. 3. Employing this monotonic behaviour and setting $r = 0$ in $H(r)$, eqn (22) readily leads to the following inequality:

$$\int_0^a p(r)H(r)r^2 \, dr > H(0) \int_0^a p(r)r^2 \, dr = \frac{h_o}{a} \frac{M}{\pi}. \tag{23}$$

Substituting the inequality of eqn (23) into (22), the upper bound solution for the rolling compliance is derived as:

$$\omega > (C_r' b_2 + C_r'')M; \quad b_2 = \frac{3[d-\ln(1+d)]}{4d^2} < 1, \quad 0 \leqslant d \leqslant 1. \tag{24}$$

Based on the upper and lower bound solutions (eqns 20 and 24), the true rolling compliance must be between

$$b_2 C_r' + C_r'' < \frac{\omega}{M} < C_r' + b_1 C_r''. \tag{25}$$

It is useful to examine the values of b_1 and b_2 which represent a measure for the range of

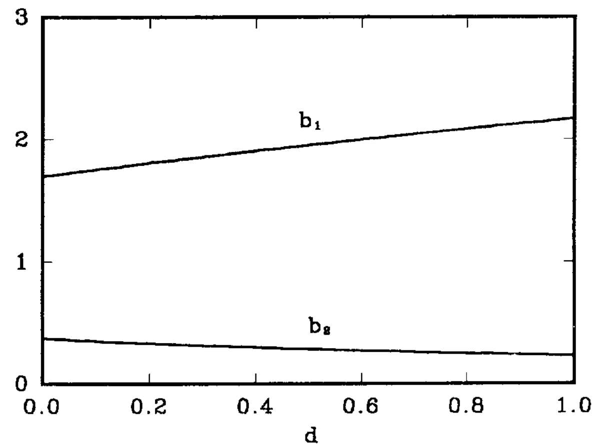


Fig. 4. Values of b_1 and b_2 with shape parameter d.

upper and lower compliance bounds. For this purpose, the values of b_1 in eqn (21) and b_2 in eqn (24) are plotted against the shape parameter, d, in Fig. 4.

3.4. *Best estimate solution*

In this section, we seek the best estimate for the rolling compliance. Two estimates are conducted. The first estimate is pursued based on the governing eqn (16), and the second estimate based on the governing eqn (22). From an alternative point of view, instead of making use of the monotonic behavior to evaluate the upper and lower bounds, we now select an approximate form for the pressure function to substitute the unknown $p(r)$ in eqn (16) or eqn (22). Thus, the relationship between force couple M and rotation ω can be estimated.

For the estimate based on eqn (16), we select the interfacial pressure $p_2(\rho)$ given in eqn (13) for the rigid binder case as the substituting pressure function to replace $p(\rho)$ in eqn (16). It can be seen that the substituting pressure function satisfy the following two limiting conditions: (1) For the rigid binder case (E_1 finite and $E_2 \to \infty$), C'_r is negligible thus this substitution yields the exact expression of the rigid binder case; and (2) for the rigid particle case ($E_1 \to \infty$ and E_2 finite), C'_r becomes dominant and C''_r is negligible, the contribution of the integral is null. This substitution yields the exact expression of the rigid particle case.

Therefore, substituting the $p(\rho)$ with $p_2(\rho)$ is a physically consistent approximation, and it leads to the following simple compliance relationship:

$$\omega = C_r M; \quad C_r = C'_r + C''_r. \tag{26}$$

When eqn (22) serves the starting point of the second estimate, we select the interfacial pressure function $p_1(\rho)$ given in eqn (11) for rigid particle case as the substituting pressure function for $p(\rho)$. The argument similar to that given in the first estimate can be stated: when C''_r is negligible (rigid particle case), the substitution yields exact solution; when C'''_r becomes negligible (rigid binder case), the contribution of the integral is null in eqn (24), thus it also yields exact solution. Therefore, substituting the $p(\rho)$ in eqn (22) with $p_1(\rho)$ is a physically consistent approximation. It is interesting to note that the second estimate yields, surprisingly, the same relationship between ω and M as the one given in first estimate (i.e., eqn (26)).

Since the rolling compliance relationship in eqn (26) satisfies the two limiting cases: rigid particle case and rigid binder case. In addition, the best estimated compliances fall in between the upper and lower bounds, i.e., the following inequalities are satisfied:

$$b_2 C'_r + C''_r < C'_r + C''_r < C'_r + b_1 C''_r. \tag{27}$$

We therefore select the rolling compliance relationship given in eqn (26) as the best estimate for this particle-binder system:

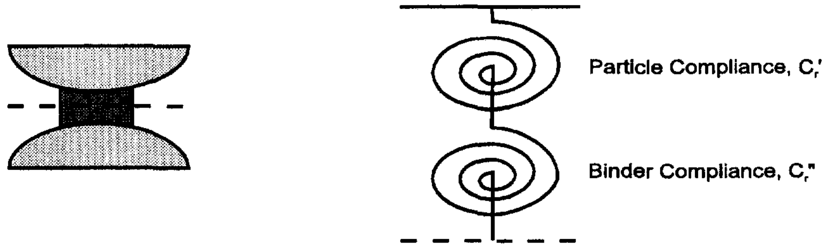


Fig. 5. Schematic plot for a serial connection of two compliances C_r' and C_r''.

$$C_r = C_r' + C_r''. \tag{28}$$

Equation (28) indicates that the rolling compliance for the particle-binder system corresponds to a serial connection of the two compliances C_r' and C_r'' as schematically shown in Fig. 5, where C_r' represents the compliance of particle and C_r'' represents the compliance of binder.

3.5. *Relationship between rolling compliance and normal compliance*

The rolling compliance is directly related to the normal compliance of the particle-binder system because they both are governed by the normal interfacial pressure. In this section, the relationship between the rolling compliance and the normal compliance is investigated.

The normal compliance of a particle-binder system has previously been derived by Zhu *et al.* (1995), given by

$$\delta = C_n P = (C_n' + C_n'')P \tag{29}$$

where δ is the relative normal approach of the two particles, P is the interfacial normal force, and C_n' and C_n'' are, respectively, the binder and particle compliances given below (see Zhu *et al.*, 1995):

$$C_n' = \frac{h_0 d}{\pi a^2 E_2 \ln(1+d)}; \quad C_n'' = \frac{1-\nu_1^2}{2aE_1} \tag{30}$$

where d is the shape parameter; a is the radius of contact area; and h_o is the thickness of binder as defined in eqn (2).

Compare eqn (26) and eqn (29), the units of rolling compliance differ from the units of normal compliance by length squared. According to the derived compliance, the ratio of rolling compliance to normal compliance for the particle-binder system is inversely proportional to the contact area. We introduce a constant Γ as follows:

$$\frac{C_r}{C_n} = \frac{C_r' + C_r''}{C_n' + C_n''} = \frac{\Gamma}{a^2}. \tag{31}$$

The values of Γ for different ratios of particle and binder moduli are shown in Fig. 6. The parameters are: $h_0 = 0.05a$, $0.1a$, and $0.5a$; $\nu_1 = 0.2$; with value of $d = 0.5$. The value of Γ ranges from 1.5 to 4.3 for various values of moduli ratio. For the case of rigid binder, the value of Γ is 1.5. The rolling compliance increases as the binder becomes softer. For very soft binder (or the rigid particle case), the value of Γ is 4.3.

4. VISCO-ELASTIC BINDER

In this section we aim to investigate the rolling compliance for two particles with a visco-elastic binder. Here, we consider two types of visco-elastic binders, namely, Maxwell and Voigt models.

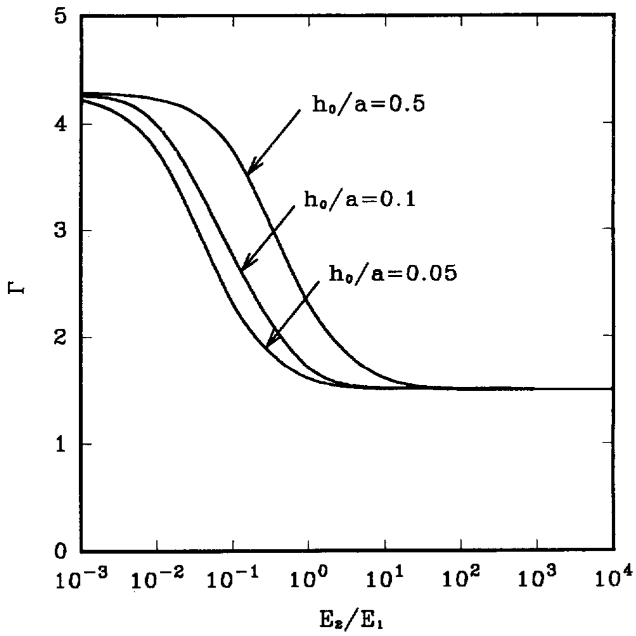


Fig. 6. Values of Γ for different ratios of particle and binder moduli.

4.1. *Maxwell model*

For Maxwell model, the normal stress-strain relationship in the thin layer of binder is given by

$$\dot{\varepsilon}_2(r,\theta,t) = \frac{1}{E_2}\dot{p}(r,\theta,t) + \frac{1}{\eta}p(r,\theta,t) \tag{32}$$

where the symbol (′) denotes the derivative with respect to t. η is the viscosity constant for the binder. In its integral representation, the stress-strain relationship reads

$$\varepsilon_2(r,\theta,t) = \frac{p^*(r,\theta,t)}{E_2} + \frac{1}{\eta}\int_0^t p^*(r,\theta,\tau)\,\mathrm{d}\tau \tag{33}$$

where $\varepsilon_2(r,\theta,t)$ denotes the normal strain, $p^*(r,\theta,t)$ denotes the normal stress.

Considering two particles bonded by a binder of Maxwell type, the governing integral equation in eqn (9) becomes:

$$\omega(t)r = h(r)\frac{p(r,t)}{E_2} + \frac{h(r)}{\eta}\int_0^t p(r,\tau)\,\mathrm{d}\tau + \frac{(1-v^2)}{\pi E_1}\int_0^{2\pi}\int_0^a \frac{p(\rho,t)\cos\phi\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{R(r,\rho,\phi)} \tag{34}$$

and

$$M(t) = \pi\int_0^{2\pi}\int_0^a p(r,t)r^2\,\mathrm{d}r. \tag{35}$$

Again, we attempt to derive an upper and a lower bound compliances for the case of Maxwell binder. In analogy to the derivation procedure for the elastic binder case, the upper bound analysis begins with multiplying $r^2/h(r)$ to eqn (34), then integrating over the range $0 \leqslant r \leqslant a$. The result is:

$$\omega(t) = C'_r\left(M(t)+\frac{E_2}{\eta}\int_0^t M(\tau)\,\mathrm{d}\tau\right)+C''_r\frac{8d^2}{3\pi[d-\ln(1+d)]}\int_0^a f(\rho)p(\rho)\rho^2\,\mathrm{d}\rho. \tag{36}$$

By utilizing the monotonic behavior of $f(\rho)$, we obtain the upper bound solution for the time dependent rolling compliance:

$$\omega(t) \leqslant (C'_r+b_1C''_r)M(t)+\frac{C'_rE_2}{\eta}\int_0^t M(\tau)\,\mathrm{d}\tau. \tag{37}$$

For the lower bound analysis, we multiply $rp_2(r)$ to eqn (34), and then integrate the equation over the ranges $0 \leqslant r \leqslant a$, which yields:

$$\omega(t) = \frac{3}{2a^3}\int_0^a\left(\frac{p(r,t)}{E_2}+\int_0^t\frac{p(r,\tau)}{\eta}\mathrm{d}\tau\right)\frac{h(r)r^2\,\mathrm{d}r}{\sqrt{a^2-r^2}}+C''_rM(t). \tag{38}$$

The lower bound can then be derived from eqn (38) by setting $r = 0$ in

$$\frac{h(r)}{\sqrt{a^2-r^2}}$$

$$\omega(t) \geqslant (C'_rb_2+C''_r)M(t)+C'_rb_2\frac{E_2}{\eta}\int_0^t M(\tau)\,\mathrm{d}\tau. \tag{39}$$

Similar to the procedure used in the last section, the first approximate solution can be obtained by replacing $p(r, t)$ in eqn (36) with the existing rigid particle solution. The second approximate solution can be obtained by replacing $p(r, t)$ in eqn (38) with the existing rigid binder solution. It turns out that both approximations lead to an identical solution. The best estimate rolling compliance is thus given by:

$$\omega(t) = (C'_r+C''_r)M(t)+C'_r\frac{E_2}{\eta}\int_0^t M(\tau)\,\mathrm{d}\tau \tag{40}$$

and its rate-dependent form:

$$\dot{\omega}(t) = (C'_r+C''_r)\dot{M}(t)+C'_r\frac{E_2}{\eta}M(t). \tag{41}$$

4.2. *Voigt model*

The normal stress-strain relationship for the Voigt binder is given by

$$p(r,\theta,t) = E_2\varepsilon_2(r,\theta,t)+\eta\dot{\varepsilon}_2(r,\theta,t) \tag{42}$$

or in its integral representation:

$$\varepsilon_2(r,\theta,t) = \int_0^t\frac{p(r,\theta,\tau)}{\eta}e^{-(E_2/\eta)(t-\tau)}\,\mathrm{d}\tau. \tag{43}$$

Accordingly, the governing integral equation for the interfacial pressure and the relative angular movement of two particles becomes

$$\omega(t) = \frac{h(r)}{\eta}\int_0^t p(r,\tau)e^{-(E_2/\eta)(t-\tau)}\,\mathrm{d}\tau + \frac{(1-v^2)}{\pi E_1}\int_0^{2\pi}\int_0^a \frac{p(\rho,t)\cos\phi\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{R(r,pho,\phi)}. \tag{44}$$

The derivation process of compliance relationships for Voigt binder is similar to that for Maxwell binder. Therefore, we list only the final results which include: an upper bound solution, a lower bound solution, and a best estimate based on physical approximation.

The upper bound solution is

$$\omega(t) \leqslant b_1 C_r'' M(t) + C_r' \frac{E_2}{\eta}\int_0^t M(\tau)e^{-(E_2/\eta)(t-\tau)}\,\mathrm{d}\tau. \tag{45}$$

The lower bound solution is

$$\omega(t) \geqslant C_r'' M(t) + b_2 C_r' \frac{E_2}{\eta}\int_0^t M(\tau)e^{-(E_2/\eta)(t-\tau)}\,\mathrm{d}\tau. \tag{46}$$

Following the similar procedure in the last section, the best estimate of rolling compliance is obtained

$$\omega(t) = C_r'' M(t) + C_r' \frac{E_2}{\eta}\int_0^t M(\tau)e^{-(E_2/\eta)(t-\tau)}\,\mathrm{d}\tau \tag{47}$$

and its rate-dependent version is

$$\dot{\omega}(t) + \frac{E_2}{\eta}\omega(t) = C_r''\dot{M}(t) + (C_r' + C_r'')\frac{E_2}{\eta}M(t). \tag{48}$$

4.3. *Equivalent spring-dashpot system*

We now view the particle-binder system as an equivalent system that consists of two rigid particles connected by springs and dashpot. Let C_r^b and C_r^p be, respectively, the spring compliance for the binder and the particle, and β be the dashpot coefficient.

The relationship between $\omega(t)$ and $M(t)$ in the equivalent spring-dashpot system of Maxwell model (i.e., Fig. 7a) is given by

$$\dot{\omega}(t) = (C_r^b + C_r^p)\dot{M}(t) + \frac{M(t)}{\beta}. \tag{49}$$

For the equivalent spring-dashpot system of Voigt model (i.e., Fig. 7b), the compliance relationship is given by

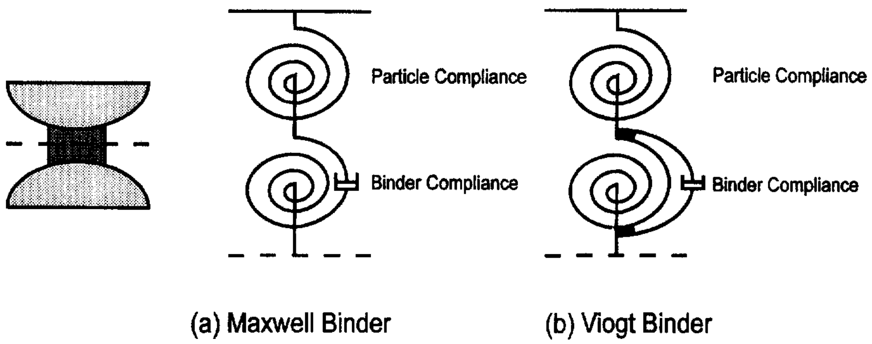


Fig. 7. Equivalent spring-dashpot systems of (a) Maxwell model and (b) Voigt model.

$$\dot{\omega}(t)+\frac{\omega(t)}{C_r^b\beta}=C_r^p\dot{M}(t)+\frac{C_r^b+C_r^p}{C_r^b\beta}M(t). \tag{50}$$

For the Maxwell model, we compare eqn (49) for the spring-dashpot system with the analytical solution (eqn (41)) for the particle-binder system. The comparison yields that $C_r^b = C_r'$ and $C_r^p \doteq C_r''$ for the spring compliances of the equivalent system and the dashpot coefficients β of the equivalent system is:

$$\beta=\frac{\eta}{E_2C_r'}=\eta a^3\frac{a}{h_0}\frac{\pi(d-\ln(1+d))}{2d^2}. \tag{51}$$

Note that the dashpot coefficient β has a unit of force-length-time while the viscosity η of the binder has the unit of force-time/length squared. The dashpot coefficient is proportional to the cubic of contact radius.

For the Voigt model, we compare eqn (50) for the spring-dashpot system with the analytical solution (eqn (48)) for the particle-binder system. The comparison yields the identical equivalent spring compliances and equivalent dashpot coefficients as those obtained from the Maxwell model. This property of model independence indicates that it is a plausible approach to use the spring and dashpot elements for the simulation of an assembly of particles.

The study presented here suggests that the compliance of a system consisting of two elastic particles with visco-elastic binder can be simulated by an equivalent spring-dash pot system. When the binder is elastic, the representation of compliance is equivalent to a two spring-element system in serial connection described in Fig. 5. When the binder is viscous of Maxwell type or Voigt type, we observe that the compliances can be schematically shown in Fig. 7a and Fig. 7b.

5. CONCLUSIONS

Rolling compliances have been derived for a system comprised of two elastic particles bonded by a thin layer of elastic or visco-elastic binder. Rolling of the two particles generates deformation of the particle-binder system and develop a force-couple at the interfacial surface. The rolling mechanism in present study is very different from that studied previously in the literature on the rolling of cylinders caused by a pull-out of metal plate. In the present problem, there is no solution available for the rolling compliance because the governing equation is of a complicated Fredholm type.

The method of analysis presented in this paper is different from the conventional approach. We do not directly solve for the exact distribution of the interfacial pressure. Instead, we alter the formulation and solve indirectly for the rolling compliance. Because it is not necessary to find the exact solution of the interfacial pressure, the approach makes it possible to yield closed-form expressions for the rolling compliance.

The method of analysis presented in this paper can be applied to solve any other integral equations of Fredholm type. The technique of employing the monotonic property of *the kernel function* is very effective in establishing upper and lower bounds for the true solution. The present method of analysis is worthwhile to be explored since integral equations of Fredholm type are common form of governing equations for a wide range of mechanical systems.

The results show that the rate-dependent compliance relationship for a system of two elastic particles bonded by a thin layer of visco-elastic binder is equivalent to a spring-dashpot system. This concept is potentially advantageous to the analysis of assemblies with a large number of particles bonded by visco-elastic binders. In the analysis of an assembly of bonded particles, the rolling compliance between particles accounts for the transmitting of force couples. It is particularly useful in the studies of stress and strain of particulate materials under shearing deformation.

Acknowledgement—This work was carried out in the course of research sponsored by the Air Force Office of Scientific Research under SETA Contract F08365-94-0020 for Han Zhu, and an AFOSR grant (No. F49620-951-0117) for Ching S. Chang. Major M. T. Chipley is the program manager. Jeff W. Rish III is the Wright Laboratory project officer.

REFERENCES

Alblas, J. B. and Kuipers, M. (1970) On the two-dimensional problem of a cylindrical stamp pressed into a thin elastic layer. *Acta Mechanica* **9,** 292.

Beesack, P. R. and Pecaric, J. E. (1985) Integral inequalities of Chebyshev type. *Journal of Mathematical Analysis Applications* **111,** 643–659.

Bentall, R. H. and Johnson, K. L. (1968) An elastic strip in plane rolling contact. *International Journal of Mechanical Sciences* **10,** 637.

Dvorkin, J., Mavko, G. and Nur, A. (1991) The effect of cementation on the elastic properties of granular material. *Mechanics of Materials* **12,** 207–217.

Dvorkin, J., Nur, A. and Yin, H. (1994) Contact laws for cemented grains: implications for grain and cement failure. *Mechanics of Materials* **18,** 351–366.

Gladwell, G. M. L. (1980) Contact problems in the classical theory of elasticity. Alphen aan den Rijn: Sijthoff and Noordhoff.

Goodman, L. E. and Keer, L. M. (1975) Influence of an elastic layer on the tangential compliance of bodies in contact. In *The Mechanics of the Contact Between Deformable Bodies*, eds A. D. de Pater and J. J. Kalker. Delft University Press, Delft, pp. 127–151.

Kanwal, Ram P. (1971) *Linear Integral Equations—Theory and Technique*. Academic Press, New York.

Keer, L. M., Kim, S. H., Eberhardt, A. W. and Vithoontien, V. (1991) Compliance of coated elastic bodies in contact. *International Journal of Solids and Structures* **27,** 681–698.

Johnson, K. L. (1985) *Contact Mechanics*. Cambridge University Press, Cambridge.

Mattewson, M. J. (1981) Axi-symmetric contact on thin compliant coatings. *Journal of the Mechanics and Physics of Solids* **29,** 89.

Meijers, P. (1968) The contact problem of a rigid cylinder on an elastic layer. *Applied Science Research* **18,** 353.

Muki, R. (1960) Asymmetric problems of the theory of elasticity for a semi-infinite solid and a thick plate. In *Progress in Solid Mechanics*, eds I. N. Sneddon and R. Hill. Vol. 1. North-Holland, Amsterdam.

Zhu, H., Chang, C. S. and Rish, J. W. (1996) Normal and tangential compliance for conforming binder contact. I: elastic binder. *International Journal of Solids and Structures* **33,** 4337–4349.

Zhu, H., Chang, C. S. and Rish, J. W. (1996) Normal and tangential compliance for conforming binder contact II: visco-elastic binder. *International Journal of Solids and Structures* **33,** 4351–4363.